1.

$$u_h = e^{\lambda x} \Rightarrow \lambda^3 = 1,\ \lambda = 1,\ \frac{-1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

$$u_h = C_1\, e^{x} + e^{\frac{-1}{2}x}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right)$$

2.

$$\lambda^2 + 4\lambda - 4 = 0 \Rightarrow \lambda = -2 \pm 2\sqrt{2}$$

$$\therefore u_h = e^{-2x}\left(C_1\, e^{2\sqrt{2}x} + C_2\, e^{-2\sqrt{2}x}\right)$$

3.

$$\lambda^2 + 5\lambda + 6 = 0 \Rightarrow \lambda = -2,\ -3$$

$$\therefore u_h = C_1\, e^{-2x} + C_2\, e^{-3x}$$

4.

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \ \therefore u_h = e^{x},\ e^{-x}$$

$$f(x) = e^{x} \text{ is the driving term}$$

$$\therefore \text{Let } u_p = C\,x\,e^{x}$$

$$\Rightarrow C\,e^{x}(2 + x) - C\,x\,e^{x} = e^{x} \ \therefore C = \frac{1}{2}$$

$$u_p = \frac{1}{2}\,x\,e^{x}$$

5.

$$\lambda^2 + 4\lambda + 4 = 0 \ \Rightarrow \lambda = -2,\ -2 \ \therefore u_h^{(1)} = e^{-2x}$$

$$\text{Let } u_h^{(2)} = v\,u_h^{(1)}$$

$$u_h^{(2)\prime} = v'\,u_h^{(1)} + v\,u_h^{(1)\prime}$$

$$u_h^{(2)\prime\prime} = v''\,u_h^{(1)} + 2\,v'\,u_h^{(1)\prime} + v\,u_h^{(1)\prime\prime}$$

$$\therefore v''\,u_h^{(1)} + 2\,v'\,u_h^{(1)\prime} + v\,u_h^{(1)\prime\prime} + 4\,v'\,u_h^{(1)} + 4\,v\,u_h^{(1)\prime} + 4\,v\,u_h^{(1)} = 0$$

$$\Rightarrow v\left(u_h^{(1)\prime\prime} + 4\,u_h^{(1)\prime} + 4\,u_h^{(1)}\right) + v''\,u_h^{(1)} + v'\left(2\,u_h^{(1)\prime} + 4\,u_h^{(1)}\right) = 0$$

$$\Rightarrow \frac{v''}{v'} = -2\,\frac{u_h^{(1)\prime}}{u_h^{(1)}} - 4 = 0$$

$$\therefore \ln v' = 0,\ v' = 1,\ v = x$$

$$\Rightarrow u_h^{(2)} = x\,e^{-2x}$$

$$\text{Let } u_p = A\,x^2 + B\,x + C$$

$$\Rightarrow A = \frac{1}{4},\ B = \frac{-1}{4},\ C = \frac{9}{8}$$

$$\therefore u = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{x^2}{4} - \frac{x}{4} + \frac{9}{8}$$

6.

$$\lambda^2 + 4 = 0,\ \lambda = \pm 2i \quad \therefore u_h = \cos 2x,\ \sin 2x$$

$$f(x) = \sin 2x \quad \therefore \text{Let } u_p = x(A \cos 2x + B \sin 2x)$$

$$\Rightarrow A = \frac{-1}{4},\ B = 0$$

$$\therefore u = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} x \cos 2x$$