Cosmology: Mid Exam

April 29, 2002

Instruction: There are six questions. Choose only four questions to answer, and write these question numbers on the top-right corner of your answer sheet. You may answer more than four questions, but only the specified questions will be marked. There is no time limit for this exam.
Equations that you may use:

Friedmann equation: $H^2 + k/a^2 = \Omega H^2 + \Lambda/3$ where $\Omega H^2 = 8\pi G\rho/3$,

Raychaudhuri equation: $\ddot{a}/a = -4\pi G(\rho + 3P)/3 + \Lambda/3$.

Constants you may use:
$G = 6.67 \times 10^{-8}\text{cm}^3\text{g}^{-1}\text{s}^{-2}$; $c = 3 \times 10^{10}\text{cm s}^{-1}$; 1 Mpc$= 3.1 \times 10^{24}$cm; $H_0 = 100h\text{km s}^{-1}\text{Mpc}^{-1}$.

1. For a Universe with $\Lambda = P = 0$:
   (a) Show that $\Omega H^2 a^3 = \Omega_0 H_0^2 a_0^3$. (4%)
   (b) Show that $a^2(\Omega - 1)H^2 = a_0^2(\Omega_0 - 1)H_0^2$. (5%)
   (c) Substitute the result of (1a) into (1b) to show that $H = H_0(1+z)\sqrt{1+\Omega_0 z}$. (10%)
   (d) Substitute the result of (1c) into (1a) to show that $\Omega = \Omega_0(1+z)/(1+\Omega_0 z)$. (6%)

2. Consider the evolution of a flat ($k = 0$) universe with $\Lambda = 0$ about the time of radiation-matter equality t_{eq} (when $\rho_{\text{r}} = \rho_{\text{m}}$). Suppose the only components in the universe are radiation and matter, i.e. $\rho = \rho_{\text{r}} + \rho_{\text{m}}$. Assume the matter component is non-relativitic and pressure-less (cold dark matter). We also define $a_{\text{eq}} = a(t_{\text{eq}}) = 1$.
   (a) We define the conformal time η such that $dt = ad\eta$. Supposing that today is matter-dominated (i.e. $\rho_0 \approx \rho_{m0}$), show that $(da/d\eta)^2 = A^2(a+1)$ where $A^2 \approx H_0^2 a_0^3$. (4%)
   (b) Integrate the above result to solve $a(\eta)$ (5%) and $t(\eta)$ (2%).
   (c) Show that $A = (2\sqrt{2}-2)/\eta_{eq}$. (2%)
   (d) Use the result of (2b) to show that $a \propto t^{1/2}$ when $\eta \ll \eta_{\text{eq}}$ and that $a \propto t^{2/3}$ when $\eta \gg \eta_{\text{eq}}$. (4%)
   (e) Given $\rho_{\text{r}0} = 8.09 \times 10^{-34}\text{g cm}^{-3}$ today, estimate z_{eq}. (4%)
   (f) Combine the results of A in (2a) and (2c) and use the result of (2e) to find t_{eq} in years. (4%)

3. Explain the following:
   (a) Definition of 1 Parsec (PC). (5%)
   (b) Olbers' paradox (including the derivation). (5%)
   (c) Hubble's Law. (5%)
   (d) The two main theories that can provide the mechanism of structure formation in the universe (just write down the names of the two theories). (5%)
   (e) The temperature at last scattering in K. (5%)

4. Consider an FRW universe:
   (a) Use the Friedmann equation and the Raychaudhuri equation to show that $d(\rho a^3) + Pd(a^3) = 0$, where ρ is the energy density, a is the scale factor, and P is the pressure. (5%)

(b) Suppose that the equation of state for a general content in the universe is $P = (\gamma - 1)\rho$, where γ is a constant. Given a_0 and ρ_0 as known, use the result of (4a) to solve ρ as a function of a. (5%)

(c) If $\gamma = 0$, $k = \Lambda = 0$, and this content is the only content in the universe, solve a as a function of t. Here we require $a = 1$ at $t = 0$. (5%)

(d) If $\gamma > 0$, $k = \Lambda = 0$, and this content is the only content in the universe, solve a as a function of t. Here we require $a = 0$ at $t = 0$. (5%)

(e) Show that if we require $a = 0$ at $t = 0$, then the solution of $a(t)$ for $\gamma < 0$ will conflict the FRW model, in which we require $a > 0$ when $t > 0$. (5%)

5. A peculiar velocity v_p is the physical velocity of an object with respect to its local comoving frame. For $k = \Lambda = 0$, consider the distance travelled by a massive particle with a non-zero peculiar velocity $v_p = dl/dt = adr/dt \neq 0$, where l is the proper distance and r is the comoving distance.

(a) Explain or prove that while the particle travels a proper distance dl, its peculiar velocity is damped by Hdl, i.e. $dv_p = -Hdl$, where $H = \dot{a}/a$. (4%)

(b) Show further that the comoving distance r the particle travels satisfies $r \propto \int dt/a^2$, and that the proper distance satisfies $l \propto \int dt/a$. (5%)

(c) Show that the following table summarizes the maximum distance the particle can travel (that is, determine whether the particle ever stops in both comoving and physical coordinates):

	Universe type	Comoving distance	Proper distance
c1 (5%)	matter-dominated	finite	∞
c2 (5%)	radiation dominated	∞	∞

(d) Suppose that the velocity of a massive neutrino is speed of light c since Big Bang until the radiation-matter equality t_{eq}, and then decays from the speed of light following the result of (a) after t_{eq}. Show that the comoving distance the massive neutrino can travel since Big Bang up to today is $5z_{eq}t_{eq}c$ in the physical coordinates today. Here z_{eq} is the redshift at t_{eq}. (Here you can assume that $a \propto t^{1/2}$ between the Big Bang and t_{eq}, and that $a \propto t^{2/3}$ between t_{eq} and today.) (6%)

6. Consider a $\Lambda = 0$ FRW model dominated by a perfect fluid with an equation of state $P = (\gamma - 1)\rho$. We define that $a' = da/d\eta$ where η is the conformal time, and that $\mathcal{H} = a'/a$.

(a) Show that the Friedmann equation and the Raychaudhuri equation can be rewritten respectively as

$$\mathcal{H}^2 + k = 8\pi G\rho a^2/3 = \Omega\mathcal{H}^2, (2\%) \tag{1}$$

$$a''/a - \mathcal{H}^2 = -4\pi G a^2 \rho(3\gamma - 2)/3. (3\%) \tag{2}$$

(b) Use this new set of equations to show that

$$\frac{d\Omega}{d\eta} = (3\gamma - 2)\mathcal{H}\Omega(\Omega - 1). (7\%) \tag{3}$$

(c) If the universe is expanding, find the range of γ for which Ω departs from unity over time. (7%)

(d) For the above range of γ, draw schematically the behavior of Ω as a function of η for $\Omega_i > 1$, $= 1$, < 1, where Ω_i is the initial value of Ω. (That is, draw three lines on the $\Omega(\eta)$ plot to show how $\Omega(t)$ change until $t \to \infty$ for $\Omega_i > 1$, $= 1$, < 1 respectively.) Suppose the universe is expanding initially. (6%)