Cosmology Example Sheet 1

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1. What are the motivations, answers, and the two main solution theories for the 'problem of structure formation'?

2. **FRW model:** Consider an isotropic 3-dimensional space (x^1, x^2, x^3) as a hypersurface embedded in a 4-dimensional Euclidean space (x^1, x^2, x^3, x^4):

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = b = \text{constant}, \tag{1}$$

The curvature of this hypersurface is positive or negative depending on the sign of b. The line element is then

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2. \tag{2}$$

(a) By using eq.(1) and differentiating it to eliminate the fictitious fourth dimention x^4 in eq.(2), rewrite dl^2 as a function of b, x^1, x^2, and x^3.

(b) Now use spherical polars

$$\begin{cases} x^1 = \rho \sin\theta \cos\phi, \\ x^2 = \rho \sin\theta \sin\phi, \\ x^3 = \rho \cos\theta, \end{cases} \tag{3}$$

to rewrite dl^2 as a function of b, ρ, θ, and ϕ.

(c) If we define

$$a = \sqrt{|b|}, \quad r = \rho/a, \quad k = \text{sign}(b) = 0, \pm 1, \quad d\Omega = d\theta^2 + \sin^2\theta d\phi^2, \tag{4}$$

then rewrite dl^2 as a function of a, r, k, and Ω. Here k is the curvature and a is the scale factor in the FRW model!

3. **Hubble's Law:** In an expanding FRW background, it is straightforward to show that the recessional velocity of two comoving observers is $v = Hl$, where $H = \dot{a}/a$ is the Hubble parameter and $l = a(t)r$ is the physical distance apart given a comoving separation r. This is often called Hubble's Law, but in fact the observationally measurable quantities are the redshift z and the luminosity distance $d_L = a(t_0)r(1+z)$.

(a) Consider a Taylor expansion of the scalefactor about $a(t_0)$ to express the redshift as

$$z = H_0(t_0 - t) + \frac{1}{2}(2 + q_0)H_0^2(t_0 - t)^2 + \ldots, \tag{5}$$

where the deceleration parameter $q = -a\ddot{a}/\dot{a}^2$.

(b) Invert this expression and eliminate t in favour of r (by considering photon propagation in the FRW spacetime) to obtain the observational Hubble Law

$$Hd_L = z + \frac{1}{2}(1 - q_0)z^2 + \ldots. \tag{6}$$

4. **FRW model:** Consider the FRW model for a closed universe:

$$dl^2 = a^2\left[\frac{dr^2}{1 - r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{7}$$

(a) What are the line elements in the three co-ordinate directions? What are the area elements in the three co-ordinate directions? What is the volume element?

(b) Suppose I draw a 'sphere' of radius r about the origin. What are its area and the volume it contains?

(c) Find the maximum area and volume of such a sphere.

(d) Suppose I have a disk of radius r in the $\theta = \pi/2$ plane. What is its area?

(e) Find the area within $r = 1$ and hence the maximum area of such a disk.

Note that there are two symmetric ranges for l as r varies from 0 to 1.

5. For a Universe with $\Lambda = P = 0$, obtain the following expressions for the values of Hubble's parameter H and the density parameter Ω at redshift z:

   (a) $H = H_0(1+z)\sqrt{1+\Omega_0 z}$.

   (b) $\Omega = \Omega_0(1+z)/(1+\Omega_0 z)$.

   (PS: For questions of this type, you can take the Friedmann eq. $H^2 + k/a^2 = \Omega H^2 + \Lambda/3$ and the Raychaudhuri eq. $\ddot{a}/a = -4\pi G(\rho + 3P)/3 + \Lambda/3$ as known results.)