Cosmology Example Sheet 3

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1. **Graviton decoupling:** Gravitons decoupled at $T \sim m_{\mathrm{Pl}}$.
   (a) Estimate their temperature today (in K) in the standard model which has $\mathcal{N} = 106.75$ for $T > 1$TeV.
   (b) Estimate their temperature today (in K) in a large GUT model which has $\mathcal{N} = 1000$ for $T > 10^{16}$GeV.
   (c) Estimate their temperature (in Mev) at $T = 1$MeV in the (a) model.
   (d) Estimate their temperature (in Mev) at $T = 1$MeV in the (b) model.

2. **Massive WIMPS:** Suppose a neutral and stable particle has a suppressed weak-scale interaction rate $\Gamma \approx 0.002 G_F^2 T^5$. Assume its mass is of order 1GeV.
   (a) Show that its decoupling temperature $T_d \approx 12$MeV.
   (b) What is the Ω_{WIMPS} today?
   (c) From the observations of CMB, cluster abundance, and Supernovae, we know that $\Omega_{\mathrm{CDM}} \approx 0.25$. So, Can WIMPS be the only cold dark matter today?

3. **Recombination and Saha's equation:** The equilibrium distributions of protons, electrons and hydrogen atoms, $p + e^- \leftrightarrow H + \gamma$, can be written in the form

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e}\left(\frac{2\pi}{m_e T}\right)^{3/2} \exp(I/T), \tag{1}$$

   where $I = 13.6$eV is the ionization energy of hydrogen.

   (a) Show that this can be re-expressed as Saha's equation:

$$\frac{1 - X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2}\eta\left(\frac{2\pi T}{m_e}\right)^{3/2} \exp(I/T), \tag{2}$$

   where the equilibrium ionization $X_e = n_e/n_B$, the baryon-to-photon ratio $\eta = n_B/n_\gamma$, with the baryon number density $n_B \approx n_H + n_p$ (ignoring helium), and assuming charge neutrality $n_e = n_p$.
   (b) In a flat universe with $\Omega_B = 0.05$ and $h = 0.65$ ($n_B = 1.13 \times 10^{-5}\Omega_B h^2 \mathrm{cm}^{-3}$), show that Saha's equation can be rewritten as $e^y = \lambda\mu y^{3/2}$ where $\lambda \approx 3.4 \times 10^{15}$ and $\mu = (1 - X_e)/X_e^2$.
   (c) A rapidly convergent way to solve this numerically is the iteration scheme $e^{y_{n+1}} = \lambda\mu y_n^{3/2}$ with $y_0 = 1$. Use this to show that when 90% of all electrons are in hydrogen atoms then the temperature is $T \approx 0.3$ eV.
   (d) By considering $\mu = 10^{-3}$ and $\mu = 10^3$, show that recombination takes place over a very narrow temperature range.
   (e) Why is the recombination temperature so much lower than the reionization energy of hydrogen?

4. **Photon decoupling and damping:** The scattering rate for photons off free electrons is given by $\Gamma_\gamma = n_e\sigma_T$, where n_e is the electron number density and $\sigma_T = (8\pi\alpha^2/m_e^2) = 6.65\times10^{-29}\mathrm{m}^2$ is the Thomson cross section.

   (a) *Mean free path:* In a flat universe, compute the photon mean free path $\lambda_\gamma = \Gamma_\gamma^{-1}$ as a function of redshift (assume $t_{dec} \gg t_{eq}$ and take $\Omega_B = 0.05$ and $h = 0.65$). (Hint: $n_e = X_e(T)n_B$, where you can use the Saha's equation in the previous question.)

   (b) *Decoupling redshift:* Show that at a redshift $1+z = 1100$, the mean free path is comparable to the horizon size, $\lambda_\gamma(t) \approx d_H(t)$. Discuss in terms of the usual definition of decoupling.

   (c) *Photon diffusion:* In a time interval Δt, a photon suffers $N = \Delta t/\lambda_\gamma(t)$ collisions. Because of this random walk, show that the mean square comoving distance travelled by a typical photon by t_{dec} is

   $$x^2 = \int_0^{t_{dec}} \frac{dt}{a^2(t)}\lambda_\gamma(t) = \frac{3t_{dec}}{5a^2(t_{dec})}\lambda_\gamma(t_{dec}), \tag{3}$$

   which today corresponds to about 30 Mpc.

   (d) *Silk damping:* Discuss the implications of this photon diffusion for (i) Doppler peaks in the CMB anisotropy power spectrum and (ii) the formation of baryonic large-scale structure.

5. **Interaction rate:** Assume the interaction rate of a particle species is a power law, $\Gamma \propto T^n$, and suppose T_d and t_d are the decoupling temerature and time when $\Gamma = H$ (with $t_d \ll t_{eq}$).

   (a) *Total decoupling:* Show that for $n \geq 3$, the average number of further interactions for each particle for the remainder of cosmic history ($t_d < t < t_0$) is less than unity.

   (b) *Thermal state problem:* Above the grand unification phase transition, interactions are mediated by massless gauge bosons with a cross-section $\sigma \approx \alpha_{\mathrm{GUT}}^2/T^2$, that is, $\Gamma \propto T$ ($n = 1$). Assuming $\alpha_{\mathrm{GUT}}^{-2} = 45$ and a minimal $\mathcal{N}_{\mathrm{GUT}} = 107$, show that thermal equilibrium cannot be maintained above $T \geq 10^{16}\mathrm{GeV}$!

6. **Nucleosynthesis parameter dependencies:**

   (a) Write down an expression which shows the explicit dependence of the nucleon decoupling temperature T_d ($\sim G_F^2 T^5$) on the number of spin degree of freedom $\mathcal{N}$.

   (b) Describe the dependence of the deuterium and ${}^3He$ 'freeze-out' temperatures T_f on the baryon-to-photon ratio η, i.e. how does T_f vary with increasing or decreasing η (hint: once ${}^2H^+$ formed, deuterium and ${}^3He$ freezed out almost immediately; check the lecture notes).

   (c) How does the abundance of ${}^4He$ change (larger or smaller) when:

      i. the weak interaction constant G_F is smaller?
      ii. there are many more neutrinos than antineutrinos or photons in the universe today?
      iii. there are many more antineutrinos than neutrinos or photons in the universe today?
      iv. there is a primordial background of gravity waves (gravitons, as in the first question)?
      v. the gravitational constant G varies with time and was larger in the past?