Cosmology: Final Exam

Jiun-Huei Proty Wu June 20, 2003

Answer all FOUR questions. The exam time is from 4:10pm to 10:10pm.

Equations that you may use:

$H^2 + k/a^2 = \Omega H^2 + \Lambda/3$ where $\Omega H^2 = 8\pi G\rho/3$; $\ddot{a}/a = -4\pi G(\rho + 3P)/3 + \Lambda/3$;
$n_i = g_i\zeta(3)T^3/\pi^2$, $\rho_i = \frac{\pi^2}{30} g_i T_i^4$, where $\zeta(3) \approx 1.2$ (relativistic Bosons);
$n_i = 3g_i\zeta(3)T^3/(4\pi^2)$, $\rho_i = \frac{7}{8}\frac{\pi^2}{30} g_i T_i^4$, where $\zeta(3) \approx 1.2$ (relativistic Fermions);
$n_i = g_i[m_i T/(2\pi)]^{3/2} \exp(-m_i/T)$, $\rho_i = m_i n_i$ (non-relativistic particles);
$H = 1.66 N^{1/2} T^2/m_{\rm Pl}$; $s = \frac{2\pi^2}{45}\mathcal{N}_s T^3$ where $\mathcal{N}_s = \sum_b g_i (T_i/T)^3 + \frac{7}{8}\sum_f g_i (T_i/T)^3$;
$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi)$; $\epsilon_V(\phi) = \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$; $\eta_V(\phi) = \frac{1}{8\pi G}\left|\frac{V''(\phi)}{V(\phi)}\right|$; $N = -8\pi G \int_{\phi_i}^{\phi_f} [V(\phi)/V'(\phi)] d\phi$

Constants you may use: $G = 6.67 \times 10^{-8}$cm^{3}g^{-1}s^{-2}; $c = 3 \times 10^{10}$cm s^{-1}; 1 Mpc$= 3.1 \times 10^{24}$cm; $H_0 = 100h$km s^{-1}Mpc^{-1}; $m_{\rm Pl} = G^{-1/2} = 1.22 \times 10^{19}$GeV; ρ_{c0}(crtitical energy density)$= 1.88 \times 10^{-29} h^2$g cm^{-3}; $G_F = 1.17 \times 10^{-5}$GeV^{-2}; 1GeV$=1.78 \times 10^{-24}g=(1.97 \times 10^{-14}cm)^{-1}=1.16 \times 10^{13}$K.

1. Consider a flat universe with $\Lambda = 0$ and a scale factor $a = \alpha t^{(n/2)}$, where t is the physical time.

   (a) First write down the comoving Hubble length $r_{\rm H}(t)$ as a function of time. What is $r_{\rm H}(t)$ in a radiation-dominated universe and in a matter-dominated universe respectively? (3%)

   (b) If the expansion rate of the universe is increasing over time, what is the range of n? (3%)

   (c) Compute the comoving distance $r(t_{\rm e}, t_0)$ that a photon can travel from the time of emission $t_{\rm e}$ to the time now t_0. (4%)

   (d) If the photons have been continuously emitted from a fixed comoving distance r_1 since $t = 0$, show that if the universe is expanding and being accelerated now and in future, then at any future time $t > t_0$ we can see the photons that were emitted only before a certain time $t_{\rm c}$. What is $t_{\rm c}$? (5%)

   (e) If $n = 1$ for $0 < t < t_2$, and $n = 4$ for $t > t_2$, what is the maximum comoving distance a photon can travel since $t = 0$? (5%)

2. **Chaotic Inflation**: Consider the Chaotic inflation $V(\phi) = \frac{1}{2}m^2\phi^2$. Assume that the field rolls towards $\phi = 0$ from the positive side, and that inflation starts at ϕ_i and ends when $\epsilon_V = 1$ or $\eta_V = 1$, whichever comes first.

   (a) Calculate the field ϕ_f at the end of inflation? (5%)

   (b) If this inflation is just enough to resolve the cosmological conundrums such as the horizon and the flatness problems, i.e. $\phi_i = \phi_{60}$, what is ϕ_i in $m_{\rm pl}$? (5%)

   (c) Under the PSRA, write down the Friedmann equation and the inflaton equation. (3%)

   (d) Given $\phi = \phi_i$ and $a = a_i$ at $t = 0$, continue to find (under PSRA) the solutions $\phi(t)$ and $a(t)$. (5%)

   (e) If we use equation of state $P = w\rho$ to describe the inflaton, calculate w at ϕ_i and ϕ_f respectively. (5%)

   (f) From the CMB result observed by the COBE satellite in 1992, we know that

   $$\delta_H \sim 10^{-5}. \tag{1}$$

   We also know that the perturbations given by inflation on the COBE scale is

   $$\delta_H \sim \sqrt{\frac{512\pi}{75}} \frac{V^{3/2}(\phi_{60})}{m_{\rm pl}^3 |V'(\phi_{60})|}. \tag{2}$$

   Combine these two results to estimate m. (5%)

(g) With this m, calculate the energy level $\rho_\phi^{1/4}$ at the beginning and the end of inflation respectively (in GeV). (5%)

(h) With this m, estimate for how long inflation last, i.e. what is $t_f - t_i$, (in seconds)? (5%)

3. Suppose a neutral and stable relativistic boson 'B' has a cross section $\sigma \approx 10^{-9} G_F^2 T^2$. It interacts with the particles that have effective number density $n = T^3$. Suppose B decouples from the thermal-equilibrium system at temperature T_d. For $T \geq T_d$, all particles in the universe are in thermal equilibrium and have an effective $\mathcal{N}_s = 100$. For $T_d > T >$ 2MeV, most particles start to annihilate so that when 2MeV$> T >$1MeV the only relativistic particles are B ($g_B = 10$), photons ($g_\gamma = 2$), neutrinos ($g_\nu = 6$), and $e^- e^+$ ($g_{e^- e^+} = 4$). Then neutrinos decouple at 1MeV, and $e^- e^+$ annihilate at 0.5MeV. B is still relativistic today.

(a) What is the decoupling temperature T_d (in GeV) of B? (4%)

(b) Given $T_\gamma = 2.73K$ and $T_\nu/T_\gamma = (4/11)^{1/3}$ today, what is the temperature of B today? (5%)

(c) What is $\bar{N}_B = n_B/s_{\rm tot}$ today, where n_B is the number density of B and $s_{\rm tot}$ is the total entropy density in the universe. (5%)

(d) What is the $\Omega_{\rm B}$ today? (5%)

(e) From the observations of CMB, cluster abundance, and Supernovae, we know that $\Omega_{\rm CDM} \approx 0.25$. Can particle B be part of the cold dark matter today and why? (3%)

4. Answer the following questions: