Cosmology Example Sheet 1

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1. **FRW model:** Consider an isotropic 3-dimensional space (x^1, x^2, x^3) as a hypersurface embedded in a 4-dimensional Euclidean space (x^1, x^2, x^3, x^4):

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = b = \text{constant}, \tag{1}$$

The curvature of this hypersurface is positive or negative depending on the sign of b. The line element is then

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2. \tag{2}$$

   (a) By using eq.(1) and differentiating it to eliminate the fictitious fourth dimention x^4 in eq.(2), rewrite dl^2 as a function of b, x^1, x^2, and x^3. (4%)

   (b) Now use spherical polars

$$\begin{cases} x^1 = \rho \sin\theta \cos\phi, \\ x^2 = \rho \sin\theta \sin\phi, \\ x^3 = \rho \cos\theta, \end{cases} \tag{3}$$

   to rewrite dl^2 as a function of b, ρ, θ, and ϕ. (4%)

   (c) If we define

$$a = \sqrt{|b|}, \quad r = \rho/a, \quad k = \text{sign}(b) = 0, \pm 1, \quad d\Omega = d\theta^2 + \sin^2\theta d\phi^2, \tag{4}$$

   then rewrite dl^2 as a function of a, r, k, and Ω. Here k is the curvature and a is the scale factor in the FRW model! (4%)

2. **Hubble's Law:** In an expanding FRW background, it is straightforward to show that the recessional velocity of two comoving observers is $v = Hl$, where $H = \dot{a}/a$ is the Hubble parameter and $l = a(t)r$ is the physical distance apart given a comoving separation r. This is often called Hubble's Law, but in fact the observationally measurable quantities are the redshift z and the luminosity distance $d_L = (1+z)a(t_0)r$.

   (a) Consider a Taylor expansion of the scalefactor about $a(t_0)$ to express the redshift as

$$z = H_0(t_0 - t) + \frac{1}{2}(2 + q_0)H_0^2(t_0 - t)^2 + \ldots, \tag{5}$$

   where the deceleration parameter $q = -a\ddot{a}/\dot{a}^2$. (4%)

   (b) Invert this expression and eliminate t in favour of r (by considering photon propagation in the FRW spacetime) to obtain the observational Hubble Law (4%)

$$H_0 d_L = z + \frac{1}{2}(1 - q_0)z^2 + \ldots. \tag{6}$$

   (c) We know that the type Ia supernovae (SNIa) have the same luminosity, independent of the redshift and cosmology. Consider two SNIa's, one in an accelerated universe and the other in a decelerated universe. Both are at the same redshift, and H_0 is the same in the two universes. Which of them looks brighter from the Earth? And why? (8%)

3. **FRW model:** Consider the FRW model for a closed universe:

$$dl^2 = a^2 \left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \tag{7}$$

   (a) What are the physical line elements dl_r, dl_θ, dl_ϕ in the three co-ordinate directions? (3%)

   (b) What are the physical area elements dA_r, dA_θ, dA_ϕ in the three directions? (3%)

   (c) Suppose I draw a 'sphere' with a comoving radius $r = 1/2$ about the origin. What are its physical area and the physical volume it contains? (4%)

   (d) What is the total physical volume of such a space? (4%)

   (e) Suppose I have a disk in the $\theta = \pi/2$ plane, and its radius is infinitely large. What is its physical area? (4%)

   (Note that there are two symmetric ranges for l as r varies from 0 to 1.)

4. **Radiation-matter transition:** Consider the evolution of a flat ($k = 0$) universe with $\Lambda = 0$ about the time of radiation-matter equality $t_{\rm eq}$ (when $\rho_{\rm r} = \rho_{\rm m}$; at this time, the conformal time is $\eta_{\rm eq}$). Suppose the only components in the universe are radiation and matter, i.e. $\rho = \rho_{\rm r} + \rho_{\rm m}$. Assume the matter component is non-relativitic and pressure-less (cold dark matter). We also define $a_{\rm eq} = a(t_{\rm eq}) = 1$.

   (a) Use the Friedmann equation to find a parametric solution for $a(\eta)$ and $t(\eta)$, where η is the conformal time ($dt = ad\eta$). (6%)

   (b) Find the asymptotic solutions of $a(t)$ for the limiting cases $\eta \ll \eta_{\rm eq}$ and $\eta \gg \eta_{\rm eq}$. (4%)

   (c) Given $\rho_{\rm r0} = 8.09 \times 10^{-34}$g cm^{-3} and $H_0 = 100h$km s^{-1}Mpc^{-1} today, estimate a_0. (4%)

   (d) Estimate $t_{\rm eq}$ in years. (4%)

   (e) Estimate $\rho_{\rm r(eq)}$ and $\rho_{\rm m(eq)}$. (4%)

5. For a Universe with $\Lambda = P = 0$:

   (a) Show that $\Omega H^2 a^3 = \Omega_0 H_0^2 a_0^3$. (2%)

   (b) Show that $a^2(\Omega - 1)H^2 = a_0^2(\Omega_0 - 1)H_0^2$. (3%)

   (c) Substitute the result of (5a) into (5b) to show that $H = H_0(1+z)\sqrt{1+\Omega_0 z}$. (4%)

   (d) Substitute the result of (5c) into (5a) to show that $\Omega = \Omega_0(1+z)/(1+\Omega_0 z)$. (3%)

6. **Einstein equations:** For a matter-dominated universe with $\Lambda = 0$, starting from the Einstein's equations

$$\begin{aligned} \dot{a}^2 + k &= 8\pi G\rho a^2/3, \\ 2a\ddot{a} + \dot{a}^2 + k &= 0, \end{aligned} \tag{8}$$

   (a) show that $k/a_0^2 = (2q_0 - 1)H_0^2$. (2%)

   (b) show that $8\pi G\rho_0/3 = 2q_0 H_0^2$. (2%)

   (c) show that $(\dot{a}/a_0)^2 = H_0^2\left[1 - 2q_0 + 2q_0(a_0/a)\right]$. (4%)

   (d) deduce that the present age of the universe is (4%)

$$t_0 = \frac{1}{H_0}\int_0^1 \left[1 - 2q_0 + \frac{2q_0}{x}\right]^{-1/2} dx. \tag{9}$$

   (e) We know that $H_0^{-1} = 9.78 \times 10^9 h^{-1} yr$. Evaluate t_0 and $t(z=1)$ for $q_0 = 0$. (4%)

   (f) Evaluate t_0 and $t(z=1)$ for $q_0 = 0.5$. (4%)