# **Cosmology** (222M6030) Example Sheet 2:
## The inflationary universe

Jiun-Huei Proty Wu

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1. **The csomological conundrums:**

   (a) The Flatness problem: Rewrite the Friedmann equation to express $\Omega^{-1}$ in terms of $k$, $a$, $\rho$, $\pi$, and $G$ only. Here $\Lambda = 0$.

   (b) (continued) For a barotropic fluid, where $\rho = \rho_0(a_0/a)^{3\gamma}$, we can obtain

   $$\left(\frac{1-\Omega}{\Omega}\right) = \left(\frac{1-\Omega_0}{\Omega_0}\right)\left(\frac{a}{a_0}\right)^{f(\gamma)}. \tag{1}$$

   What is the function $f(\gamma)$?

   (c) What is the value of $\gamma$ in a matter-dominated universe?

   (d) (continued) Assume that from the last-scattering epoch ($z_{\rm ls} \approx 1000$) to today, the universe is always matter-dominated. If $\Omega_0 = 0.01$ today, estimate $\Omega$ at last scattering.

   (e) (continued) If $\Omega_0 = 100$ today, estimate $\Omega$ at last scattering.

   (f) The horizon problem: A ray of light is emitted from a point A at time $t = 0$. It reaches a point B at the time of decoupling $t_{\rm dec}$ (178,000 $h^{-1}$ years). Assuming the universe remained radiation dominated from $t = 0$ to $t_{\rm dec}$, calculate the physical separation (in $h^{-1}$Mpc) of A and B at $t_{\rm dec}$. A and B are both comoving objects.

   (g) (continued) If the decoupling occurs at $z_{\rm dec} \approx 1000$, what is the physical separation of A and B today (in $h^{-1}$Mpc)?

   (h) (continued) Imagine A and B to be at two locations on the last scattering surface, whose radius is about 6000 $h^{-1}$Mpc today. What would be their apparent angular separation in a flat universe (in degrees)? (note: 1 pc = 3.26 light years.)

2. **Scalar fields**: A free scalar field is one for which the potential is identically zero, i.e. $V(\phi) = 0$.

   (a) If we describe a free scalar field as a perfect fluid, $P = w\rho$, what is the value of $w$?

   (b) Show that this free scalar field acts like a barotropic fluid, which satisfies $\rho = \rho_0(a_0/a)^{3\gamma}$, and find $\gamma$.

   (c) If the universe is flat and homogeneous, use Friedmann equation to find the solution $a(t)$.

3. **Chaotic Inflation**: In addition to $V(\phi) = \frac{1}{2}m^2\phi^2$, Chaotic inflation also refers to models based on other simple potentials. One example is $V(\phi) = \frac{\lambda}{4}\phi^4$, which is also called the self-coupling inflation. Assume that the field rolls towards $\phi = 0$ from the positive side, and that inflation starts at $\phi_i$ and ends when $\epsilon_V = 1$.

   (a) Calculate the value of $\phi$ where each of the slow-roll conditions first break down. Do they break down at the same place?

(b) Calculate the field $\phi_f$ at the end of inflation?

(c) Use this result to further calculate the total number of $e$-foldings $N(\phi_i)$ of the inflation.

(d) If this inflation is just enough to resolve the cosmological conundrums such as the horizon and the flatness problems, i.e. $\phi_i = \phi_{60}$, what is $\phi_i$ in $m_{\rm pl}$ and in GeV respectively?

(e) Given this $\phi_i$, calculate $\epsilon_V$ and $\eta_V$ at $\phi_i$.

(f) Given $\phi = \phi_i$ and $a = a_i$ at $t = 0$. Under the PSRA, simplify the equations of motion (i.e. eqns (8.7) and (8.8) in the lecture notes) and then find the solutions $\phi(t)$ and $a(t)$.

(g) Expand the solution for $a(t)$ at small $t$ to demonstrate that the inflation is approximately exponential at the initial stage.

(h) Use the results in (3b), (3d), (3f) to calculate the time $t_f$ when inflation ends.

(i) Use the results in (3b), (3d), (3f) to calculate $\rho_\phi$ at $\phi_i$ and $\phi_f$ respectively.

(j) If we use equation of state $P = w\rho$ to describe the inflaton, calculate $w$ at $\phi_i$ and $\phi_f$ respectively.

(k) From the CMB result observed by the COBE satellite in 1992, we know that

$$\delta_H \sim 10^{-5}. \tag{2}$$

We also know that the perturbations given by inflation on the COBE scale is

$$\delta_H \sim \sqrt{\frac{512\pi}{75}}\frac{V^{3/2}(\phi_{60})}{m_{\rm pl}^3\,|V'(\phi_{60})|}. \tag{3}$$

Combine these two results to estimate $\lambda$.

(l) With this observationally constrained $\lambda$, use the result in (3h) to estimate $t_f$ in seconds.

(m) With this $\lambda$, use the results in (3i) to further calculate the energy level $\rho_\phi^{1/4}$ at the beginning and the end of inflation respectively (in GeV).

(n) Use the results in (3e) to estimate the spectral indices $n$ and $n_T$ of the scalar and the tensor perturbations respectively. (You may use the formulae in the lecture notes.)